PD Dr. Maksim Zhykhovich

Summer Term 2025
25.07.2025

Algebraic theory of quadratic forms and Kaplansky's problem

Exam

Family Name: ______________ First name: ______________

Student ID: ______________ Term: ______________

Degree course: Bachelor, PO ❏ 2011 ❏ 2015 ❏ 2021 Master, PO ❏ 2011 ❏ 2021

Lehramt Gymnasium: ❏ modularisiert ❏ nicht modularisiert

❏ Diplom ❏ Other: ______________

Major subject: ❏ Mathematik ❏ Wirtschaftsm. ❏ Inf. ❏ Phys. ❏ Stat. ❏ ______

Minor subject: ❏ Mathematik ❏ Wirtschaftsm. ❏ Inf. ❏ Phys. ❏ Stat. ❏ ______

Credit Points to be used for ❏ Hauptfach ❏ Nebenfach (Bachelor / Master)

Please switch off your mobile phone and do not place it on the table; place your identity and student ID cards on the table so that they are clearly visible.

Please do not write with the colours red or green. Write **on every page** your **family name and your first name**.
Write your solutions on the page marked with the appropriate problem number. If you run out of space, use the empty pages at the end of the examination paper ensuring that each problem is clearly marked.

Good luck!

1	2	3	4	$\sum$
/14	/11	/12	/16	/53

Name: ____________________

Problem 1. **[14 Points]**

1) Show that the following quadratic forms are isometric [2+4=6 Points]

(a) $\langle 1, 1, 1, -1\rangle \simeq \langle 1, 2, 3, -4\rangle$ over $\mathbb{R}$,

(b) $\langle 1, 1, -5\rangle \simeq \langle 2, -2, 5\rangle$ over $\mathbb{Q}$.

2) Let q be a 6-dimensional quadratic form over a field F with $\det q = -1 \in F^\times / F^{\times 2}$.

[3+5=8 Points]

(a) Show that if $i(q) > 1$ then $i(q) = 3$.

(b) Assume that q is anisotropic. Show that $i_1(q) = 1$, where $i_1(q)$ denotes the first Witt index of q (it equals to the minimal non-zero value of $i(q_E)$, where E runs over all field extensions of F).

Name: ____________________

Problem 2. **[6+5=11 Points]**

1) Denote by $\mathbb{R}_9$ the functon field of the 9-dimensional quadratic form $9\langle 1\rangle$ over $\mathbb{R}$.
Find the Witt index of the form $10\langle 1\rangle$ over $\mathbb{R}_9$.

2) Let $n \geq 2$. Show that the element $x_1^2 + ... + x_n^2$ is not a sum of $n-1$ squares in $\mathbb{R}(x_1, ..., x_n)$.

Name: ____________________

Problem 3. **[5 + 7 = 12 Points]**

Let F be a field. Let π be an n-fold Pfister form over F and let φ be an m-fold Pfister form over F with $m > n > 0$.
Assume that $\pi \subseteq \varphi$ (π is a subform of φ).

1) Show that $\varphi \simeq \pi \otimes \rho$ for some quadratic form ρ over F.
Hint: Use function fields of quadratic forms.

2) Show that $\varphi \simeq \pi \otimes \pi'$ for some Pfister form π'.
Hint: Use 1) and first consider the case $m - n = 1$. Then proceed by induction on $m - n$.

Name: ______________________________

Problem 4. **[2+4+6+4=16 Points]**

Let $n \geq 2$ and let $q = \langle a_1, ..., a_n \rangle$ be a diagonal quadratic form with $a_1, ..., a_n \in F^\times$. We define $s(q)$ to be the tensor product over F of the following quaternion algebras

$$\bigotimes_{1 \leq i < j \leq n} \left(\frac{a_i, a_j}{F} \right) .$$

1) Show that $s(q)$ is a central simple algebra over F.

2) Let $q = \langle a, b \rangle$ and $q' = \langle c, d \rangle$, where $a, b, c, d \in F^\times$. Assume that $q \simeq q'$.
Show $s(q)$ and $s(q')$ are isomorphic as F-algebras, that is

$$s(q) = \left(\frac{a, b}{F} \right) \simeq \left(\frac{c, d}{F} \right) = s(q') .$$

3) Let $n \geq 2$. Let $q = \langle a, b, a_3, ..., a_n \rangle$ and $q' = \langle c, d, a_3, ..., a_n \rangle$ be two diagonal quadratic forms over F, where $a, b, c, d, a_3, ..., a_n \in F^\times$.
Assume $q \simeq q'$. Show that $s(q)$ and $s(q')$ are isomorphic as F-algebras.

4) Let q and q' be two isometric diagonal quadratic forms over F. Show that $s(q)$ and $s(q')$ are isomorphic as F-algebras.

Name: ______________________________

Name: ____

Name: ______________________________